BETA-DISPERSION OF LOW-FREQUENCY ROSSBY WAVES

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ABSTRACT


An investigation is made into the dispersion of oceanic internal Rossby waves at annual and semi-annual frequencies. Turning of the group velocity vector due to latitudinal variations in the radius of deformation cannot be neglected, particularly in basins as large as the Pacific. This turning allows disturbances to propagate from high latitudes into the equatorial zone and distorts the solutions in the western part of the basin. For no mean flow, and a coastline aligned north—south, an almost exact focus of wave energy is found very close to the equator at a distance of just under \( \pi c/4\omega \) from the eastern boundary, where \( c \) is the eigenspeed of a high-frequency internal wave mode, and \( \omega \) is the angular frequency of the low-frequency wave being studied. The focus depends on a long meridional wavelength excited at the coast, and a frequency small compared with \( c/a \), where \( a \) is the radius of the Earth. For the lowest baroclinic mode and waves of annual period, this distance is about 12 000 km. Equivalence of the ray theory and the theory of equatorial meridional modes is demonstrated for the simple cases where the latter applies.

The effects of mean currents and irregular coastlines are examined. Barotropic mean currents may change the turning latitude and ray shapes, inducing critical layers and enhancing reflection. Baroclinic mean currents are seen to affect the rays by simply changing the speed in proportion to the depth of the thermocline. As long as the mean currents are geostrophically balanced, no "effective beta" term from variations in the thermocline depth appears, in contrast to the topographic Rossby wave problem.

1. INTRODUCTION

In the atmosphere there is a very pronounced annual weather cycle. Since there is weak coupling between the atmosphere and the oceans, the oceans

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must also exhibit annual variations. An extreme example is the reversal of the Somali current, but this is associated with the relatively sudden onset of the Southwest Monsoon (Lighthill, 1969) rather than with the more typical gradual weather changes. A natural way of analyzing unsteady oceanic currents is to decompose the motion into two parts: the localized forced response for a hypothetical unbounded ocean, and free waves emanating from the ocean boundary. To date there has not been clear observational evidence of such an annual (or semi-annual) signal propagating in the ocean (Roden, 1977). The question towards which the present work is directed is whether there are particular regions of the ocean where the annual signal can be expected to be particularly intense.

On a $\beta$-plane the zonal velocity satisfies

$$\frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u - \left( f_0^2/c^2 \right) \frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = 0 \quad (1.1)$$

where $f_0$ and $\beta$ are the constant local values of the Coriolis parameter and of its meridional gradient. This equation is valid for any given vertical mode. For different modes the value of $c$ (eigenspeed of a high-frequency internal wave) is different. For the first baroclinic mode $c$ is typically $3 \text{ m s}^{-1}$, for the second about $1 \text{ m s}^{-1}$.

If we consider low-frequency long Rossby waves such as might be generated by the wind at an eastern boundary and propagate westwards, (1.1) can be simplified to

$$\frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} u - \left( l^2 + f_0^2/c^2 \right) \frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = 0 \quad (1.2)$$

where $l$ is the meridional wavenumber. This equation is analogous to that used by Anderson and Gill (1975) to examine the spin-up of a stratified ocean. In their paper the meridional scale was imposed by the wind. At mid-latitudes this leads to estimates of $l^2 \approx 10^{-12} \text{ m}^{-2}$ and $f_0^2/c^2 \approx 10^{-9} \text{ m}^{-2}$. Thus the neglect of the $l^2$ term in (1.2) appears to be well founded except possibly at very low latitudes, where in any event the oceanic motions are best described by expanding the north–south variation in Hermite functions rather than in sinusoids (Blandford, 1966). Although the above argument is quantitatively correct for the spin-up problem considered by Anderson and Gill, it is not correct for periodic forcing. Even at periods as long as one year, there is a significant error. One can give a simple qualitative argument to show this. With the $(\partial/\partial t)(\partial^2/\partial y^2) u$ term neglected, (1.1) can be solved independently for each latitude. Thus, it is permissible to relax the $\beta$-plane approximation and to take $f_0$ to be the actual non-constant Coriolis parameter. It follows that the westerly wave speed is a function of latitude. Figure 1 shows the resulting wave pattern for low-frequency waves originating from a meridional boundary. The full lines give the positive peaks in the phase (say the summer signal) and the broken lines the negative peaks half a cycle later. Although
the north-to-south wavelength may have been large initially, it quickly becomes much shorter because of the spatial inhomogeneity of the solution: the tendency for low-latitude waves to travel faster than high-latitude waves.

This can be quantified as follows. After \( n \) years the solid wave front will be at the position \((x_0, y_0)\), given by \( P \) in Fig. 1, where \( x_0 \) is given by \( \beta n c^2/f_0^2 \).

Half a year earlier and later the wavefronts at this longitude \( x_0 \) will be at latitudes \( \theta_1, \theta_2 \) marked by points A and B, where

\[
A: (n + 1/2) c^2/f_1^2 = nc^2/f_0^2, \text{ i.e. } \sin \theta_1 = (1 + 1/2n)^{1/2} \sin \theta_0 \\
B: (n - 1/2) c^2/f_2^2 = nc^2/f_0^2, \text{ i.e. } \sin \theta_2 = (1 - 1/2n)^{1/2} \sin \theta_0
\] (1.3)

The spatial separation of the points is \((y_1 - y_2)\) and thus the effective north-to-south wavenumber is

\[
l = 2/R(\theta_1 - \theta_2)
\] (1.4)

where \( R \) is the radius of the Earth. For moderately large \( n \) the general expression in (1.4) can be simplified to

\[
l = 4\pi n/R \tan \theta_0
\] (1.5)

At \( 20^\circ \)N say, \( f^2/c^2 \approx 0.6 \times 10^{-10} \) and after two years \( l^2 \) is \( 0.6 \times 10^{-10} \text{ m}^{-2} \). Thus, the two terms are about equal. This means the dispersion of these waves
is important and the conventional method of using (1.1), i.e. of applying it as a constant-coefficient equation at any given latitude (White, 1977), is false.

Instead one must allow for dispersion of these Rossby waves and this will now be done in the following two sections using a ray theory. It will then be extended to cases where the coast is not aligned exactly north—south, but at an angle. The next section compares the ray theory results with those from modal theory. The final section examines the effects of mean currents. In the theory that follows there is no need for the period to be annual. Any low-frequency waves, from a few weeks to several years, are covered. Except at low latitudes, however, free Rossby waves will exist only for the very low frequencies.

2. RAY THEORY

Equation 1.1 is not valid near the equator. It is a mid-latitude approximation. However, a similar equation for the meridional velocity \( v \), valid at the equator is obtainable (Rattray, 1964)

\[
\frac{3}{t^3} \frac{\partial^3}{\partial t^3} v/c^2 + \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v - \left( \beta^2 y^2/c^2 \right) \frac{\partial v}{\partial t} + \beta \frac{\partial v}{\partial x} = 0
\]

(2.1)

This equation does not contain information on the equatorial Kelvin wave for which \( v \) is identically zero. However, since we are interested in westward-travelling waves and the Kelvin wave goes eastward, this omission is not of consequence to this analysis. It is of course possible for the long Rossby waves impinging upon a meridional boundary to excite a reflected equatorial Kelvin wave.

In obtaining (2.1) the equatorial \( \beta \)-plane approximation has been made. This makes the analysis simpler because it removes the metric terms present in the spherical polar representation. Also, we shall discard the \( \partial^3 v/\partial t^3 \) term and consider only low-frequency waves. These two approximations have been studied by Longuet-Higgins (1965). The associated errors are quantitative only; none of the basic results are altered.

Before examining the behavior of low-frequency Rossby waves in the cases of interest here, a review of ray theory as applied to Rossby waves will be given. Blandford (1966) has examined ray theory for mixed-gravity Rossby waves on the equatorial \( \beta \)-plane. For the low-frequency cases considered here, many of the expressions for the ray paths and wavenumber evolution are simplified due to the reduced order of the dispersion relation.

The method of dealing with dispersive equations is by the WKB technique, which is valid provided that the scale of the spatial variations is large compared with the wavelength of the waves. This is true for the waves discussed here, since the scale of the variation in \( f \) is global but the scale of the waves is only a few degrees, and may be checked a posteriori.

At any locality the waves are assumed to be approximately sinusoidal:

\[
v = \text{Re} \{ A(x, y, t) \exp[i \phi(x, y, t)] \}\]
where the wave amplitude $A$ varies slowly in space and time in comparison with variations in the phase $\phi$. The total wavenumbers $k$, $l$ and frequency $\omega$ are defined as derivatives of the phase:

$$k = \partial \phi / \partial x, \quad l = \partial \phi / \partial y, \quad w = -\partial \phi / \partial t$$

As long as $(1/A) \partial^2 A / \partial x^2 << k^2 + l^2$, $(1/A) \partial^2 A / \partial y^2 << k^2 + l^2$, $(1/A) \partial^2 A / \partial t^2 << \omega^2$, then $k$, $l$ and $\omega$ obey the dispersion relation

$$\omega = \Omega(x, y, t; k, l) = -\beta k/(k^2 + l^2 + \beta^2 y^2/c^2)$$ (2.2)

Mathematically, the dispersion relation can be interpreted as a non linear first-order differential equation for the phase of the waves. The group-velocity or ray paths

$$dx/dt = \partial \omega / \partial k \tag{2.3}$$
$$dy/dt = \partial \omega / \partial l \tag{2.4}$$

are the bi-characteristics along which the partial differential equation 2.2 can be replaced by ordinary differential equations for $\omega$, $k$, $l$ (Whitham, 1960):

$$dk/dt = \partial \Omega / \partial x \tag{2.5}$$
$$dl/dt = \partial \Omega / \partial y \tag{2.6}$$
$$d\omega/dt = \partial \Omega / \partial t \tag{2.7}$$

Physically, the ray paths are important because the wave energy (or more precisely the wave action) is carried along the rays.

Here the propagating medium is time-independent and does not vary in the east—west direction. This has the consequence that along a ray the frequency $\omega$ and the east—west wavenumber $k$ are both conserved. For the dispersion relation (2.2), (2.3)—(2.7) become

$$dx/dt = \omega/k + 2\omega^2/\beta \tag{2.8}$$
$$dy/dt = (2\omega^2/\beta k) l \tag{2.9}$$
$$dl/dt = (2\omega^2/\beta k c^2) y \tag{2.10}$$

The solutions to these equations are

$$k = k_0, \quad \omega = \omega_0 \tag{2.11}$$
$$l = \left(\frac{\beta k}{\omega} - k^2\right)^{1/2} \sin\left(\frac{2\omega^2}{kc} t + \Theta_0\right) \tag{2.12}$$
$$x = x_0 + \left(\frac{\omega}{k} + \frac{2\omega^2}{\beta}\right) t \tag{2.13}$$
$$y = \frac{c}{\beta} \left(\frac{\beta k}{\omega} - k^2\right)^{1/2} \cos\left(\frac{2\omega^2}{kc} t + \Theta_0\right) \tag{2.14}$$

where $x_0$ and $\Theta_0$ are constants of integration.
For completeness we record that along ray paths the phase of the waves is given by

$$\phi = \phi_0 - \left(1 - \frac{\omega k}{\beta}\right) \omega t - \frac{k c}{4 \omega} \left(1 + \frac{\omega k}{\beta}\right) \sin \left(\frac{4 \omega^2 t}{k c} + \Theta_0\right) + \frac{n \pi}{2}$$

(2.15)

where $n$ is the number of times the ray has gone through a caustic, and $\phi_0$ is a further constant of integration to be determined. For a stationary observer, the phase decreases at the rate $-\omega$. Thus, at first sight it might seem surprising that an observer moving with the rays experiences on average a more rapid decrease in phase. The reason for this is that the phase velocity can be inclined at greater than a right angle to the direction of the group velocity (Fig. 2) which implies an enhanced rate of crossing of phase lines for an observer moving with the rays. This effect is most pronounced at the equator and the phase decreases at twice the imposed frequency, i.e. at the rate $-2\omega$. When a ray is at its furthest distance from the equator, the phase and group velocities are aligned and the phase decreases at its minimum rate, $2\omega^2 k/\beta$ ($k$ is negative).

3. WAVES FROM A MERIDIONAL BOUNDARY

The results of Section 2 will now be applied to obtain the dispersion of Rossby waves generated from a plane eastern boundary aligned along $x = 0$. If

$$\Theta = -2\omega^2 t/k c$$

$$y^2_t = \left(c^2/\beta^2\right)\left(-\beta k/\omega - k^2\right)$$

(3.2a)

$$y^2_0 = l_0^2 c^2/\beta^2$$

(3.2b)

where the subscript 0 refers to the initial values of $y$ and $l$, the ray equations
2.7—2.11 become

\[ x = x_0 - \frac{c}{2\omega} \left( 1 - \frac{4\omega^2}{c^2} y_T^2 \right) \Theta \]  

(3.3)

\[ y = y_T \cos(\Theta + \Theta_0) \]  

(3.4)

\[ l = \frac{\beta}{c} y_T \sin(\Theta + \Theta_0) \]  

(3.5)

\[ \Theta_0 = \begin{cases} 
\cos^{-1}(y_0/|y_T|) & \text{if } l_0 > 0 \\
-\cos^{-1}(y_0/|y_T|) & \text{if } l_0 < 0 
\end{cases} \]  

(3.6)

where \( y_T \) is the maximum latitude to which the Rossby wave energy along this particular ray may penetrate. The ray path shape depends on only \( y_T \) and \( \Theta \). \( x_0 \) and \( \Theta_0 \) fix the phase of the sinusoidal ray path in relation to the coast.

3.1. Zero initial meridional wavenumber

For simplicity, we consider uniform wave amplitude and phase excited along \( x = 0 \). These restrictions will be subsequently relaxed. At \( t = 0 \) we require that the rays leave the boundary with \( l_0 = 0 \). Therefore, \( \Theta_0 = 0 \), and \( y_T = y_0 \). The ray path equations are

\[ x = -(c/2\omega) \left( 1 - 4\omega^2 y_0^2/c^2 \right) \Theta \]  

(3.7)

\[ y = y_0 \cos \Theta \]  

(3.8)

Figures 3 and 4 show the ray paths for annual and semi-annual periods.

Fig. 3. Ray paths for annual period waves on the equatorial beta plane. The tick marks indicate the time in years along each path. \( l_0 = 0 \).
Fig. 4. Ray paths for the semi-annual period waves. $l_0 = 0$.

(1) The rays are as predicted by the arguments of Section 1, not straight lines running east—west, but sinusoidal, curving towards the equator initially. This implies that the energy is channelled towards the equator.

(2) There is an area where the rays are packed together (a caustic) and an area where rays from the same side of the equator do not penetrate (a shadow zone). The latter feature is of great importance for it indicates that there are areas in the ocean where Rossby waves from the east will not penetrate (except for slow-moving waves from the far side of the equator).

(3) In the semi-annual case, the rays exist only up to $30^\circ$N. The rays for $35^\circ$N and poleward are not drawn because there are no free Rossby waves in this latitude band for this frequency. For the annual signal, free Rossby waves exist at all latitudes plotted.

When $\Theta = \pi/2$, there is a position on the equator at

$$x = -\pi c/4\omega$$  \hspace{1cm} (3.9)

which is an almost exact focus for the rays. Thus, close to this point the waves will be exceptionally intense. For the annual signal the position is nearly 12 000 km west of the ocean boundary.

At the instant of departure from the eastern boundary $l$ is zero, but this does not cause any difficulty with the ray approach. The meridional scale of the wave is larger than the scale of the variation of $f$, but there remains a non-zero zonal wavenumber and $\partial^2 A/\partial y^2$ remains smaller than $k^2 A$, the necessary condition for validity of the ray theory. The possible exception to this is for
a ray starting with \( l = 0 \), near the maximum turning latitude for that frequency. Thus in Fig. 4 rays starting close to where the caustic meets the coast will not be correctly handled but these are of no great physical significance. For other rays subsequently intersecting the caustic, ray theory can be shown to work satisfactorily (Peregrine and Smith, 1979). Blandford (1966) considered a problem where the intersection of the rays with the caustic was at a turning latitude. In this case, as noted above, ray theory fails and Airy matching is required. For the problem considered here, this is not so.

The condition for adjacent rays to cross is that

\[ \frac{\partial (x, y)}{\partial (y_0, \Theta)} = 0 \]

Evaluating the Jacobian, we find that along the caustic \( \Theta \) and \( y_0 \) are related by

\[ \Theta \tan \Theta + 1 = \frac{c^2}{4\omega^2}y_0^2 \]  

(3.10)

Thus, eliminating \( y_0 \) in favour of \( \Theta \), the caustic can be represented

\[ x = -\frac{c}{2\omega} \Theta^{3/2} (\tan \Theta)^{1/2} (1 + \Theta \tan \Theta)^{-1/2} \]  

(3.11)

\[ y = \pm \frac{c}{2\omega} (1 + \Theta \tan \Theta)^{-1/2} \cos \Theta \]  

(3.12)

where \( \Theta \) ranges between 0 and \( \pi/2 \).

For small \( \Theta \) the asymptotic approximations are

\[ x = -(c/2\omega) \Theta^2 + ... , \quad y \mp c/2\omega = \mp (c/2\omega) \Theta^2 + ... \]

Thus, at the critical latitude the caustic is at an angle \( \pi/4 \) to the meridional boundary. For \( \Theta \) close to \( \pi/2 \) the local description of the caustic curves is

\[ x + \frac{\pi c}{4\omega} = \frac{3c}{4\omega} \left( \frac{\pi}{2} - \Theta \right) + ... , \]

\[ y = \pm \frac{c}{\omega (2\pi)^{1/2}} \left( \frac{\pi}{2} - \Theta \right)^{3/2} + ... \]

Thus, the two branches of the caustic either side of the equator meet at a cusp.

Peregrine and Smith (1979, Appendix B) show that although the ray solution is not itself valid at such singularities, it does provide all the information needed to construct a uniformly valid description of the locally intense and rapidly varying wave amplitude.

For an observer moving with the rays, the phase is constrained to be zero at \( t = 0 \) when the ray initiates from the boundary. Since \( t_0 = 0 \), it follows from (2.15) that \( \phi_0 = 0 \) and \( \phi \) is given by

\[ \phi = -\left( 1 - \frac{\omega k}{\beta} \right) \omega t - \frac{kc}{4\omega} \left( 1 + \frac{\omega k}{\beta} \right) \sin \left( -\frac{4\omega^2 t}{kc} \right) + \frac{n\pi}{2} \]  

(3.13)
If we wish to construct phase contours at a given time, then we must allow for the fact that when a ray originally left the meridional boundary, the phase would have been non-zero. Formally, this can be achieved by adding $\omega t$ to

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**Fig. 5.** Above: phase contours before the caustic for the rays in Fig. 3. The rays from 15, 30 and 45°N are indicated. Below: phase contours for rays beyond the caustic, showing rays which originate at 50 and 60°N.
the result (3.13), and by reinterpreting the parameter \( t \) as a generalized distance coordinate. In terms of \( y_0 \) and \( \Theta \) the phase at a fixed time is given by

\[
\Theta = \frac{c\beta}{4\omega^2} \left\{ \Theta \left[ 1 - \frac{2y_0^2\omega^2}{c^2} - \left( 1 - \frac{4y_0^2\omega^2}{c^2} \right)^{1/2} \right] - \frac{y_0^2\omega^2}{c^2} \sin 2\Theta \right\} + \frac{n\pi}{2} \tag{3.14}
\]

This result, together with the ray equations 3.7—3.8, enables us to calculate the phase contours (Fig. 5). Because there can be more than one ray passing through a given physical point, there is a corresponding multiplicity of values for the phase. This can be resolved by considering separately rays before and after the caustic.

The primary branch of the phase function, corresponding to rays which have not yet reached the caustic, is in close agreement with the simplest theory for Rossby waves as discussed in Section 1. Away from the coastline there is a systematic increase in \( \phi \). However, due to the presence of the caustic, the phase does not increase indefinitely, and there is a maximum value of \( \phi \) at about 40°N. The second branch of the phase function exhibits the decrease in phase which was explained at the end of the previous section (Fig. 2). For this second branch the phase contours are very much more closely packed than for the first branch (the contour interval is \( 10\pi \)). Thus, if our analysis were to be extended to include frictional effects, wave action would not be conserved along ray tubes and, moreover, rays beyond the caustic would be most seriously damped. Hence, it is only the first branch of the phase function that could meaningfully be compared with real oceanic data or with numerical simulations.

3.2. Non-zero initial meridional wavenumber

In this case \( x_0 \neq 0 \), but otherwise (3.1—3.6) apply. The form (3.2b) is most useful in determining \( y_0 \), and the dispersion relation must be used to determine \( k \).

If the forcing along a meridional coast is a standing wave, the solution will consist of two sets of waves, one for \( l_0 > 0 \) and one for \( l_0 = -l_0 \). In this case the two rays will diverge, one having a northward component and one a southward component. The standing wave pattern in \( y \) seen at the coast will disappear as these rays diverge.

In addition to this divergent response, the non-zero initial meridional wavenumber changes the shadow zone concept, which relies on organized phase structure at \( x = x_0 \). However, the approximation of no meridional structure is good, provided that the wavelength is long compared with \( L_e^2/y_0 \), where \( L_e \) is the equatorial radius of deformation, \( (c/\beta)^{1/2} \). For the internal waves considered here, \( L_e \) is of the order of 3°, so that structure long compared with 1° at \( y_0 = 10° \), or compared with 2° at \( y_0 = 5° \) may be considered "long" for the purposes of shadow zones.

It is also noted that from any starting latitude, the waves with the longest
zonal wavelength will have $l_0 = 0$. If any wave energy does enter the shadow zone by virtue of a non-zero initial wavenumber, it should have a shorter wave structure.

4. SLOPING COASTLINES

In the previous section coast was aligned north—south. This restriction will be relaxed here. The WKB description of the waves is valid for coastlines of arbitrary shape provided that the variations are not too rapid. However, for simplicity, we will only discuss straight coastlines.

The analysis is similar to Section 3, except that now it is not the meridional structure which is given but the along-shore wavenumber. If $l_s$ is the projection of the wavenumber vector along the coast, then the initial wavenumber $k = (k_0, l_0)$ must satisfy both the dispersion relation (2.2) and the condition

$$k \cdot e_s = l_s$$

(4.1)

where $e_s$ is a unit vector along the coast. The nature of the solutions can be best seen by using the wavenumber diagram of Longuet-Higgins (1965). This is drawn schematically in Fig. 6, with a coast aligned at angle $\alpha$ to the meridian.
included on the diagram. The normal to the coast from a point $l_s$ away from the origin intersects the wavenumber circle at two points. We are interested in the right-most point (i.e. with group velocity away from the coastline). The coordinates in wavenumber space give the starting values of $k_0$, $l_0$ for the rays such that the wave structure along the coast is in agreement with our choice of $l_s$, i.e.

$$l_0 = l_s / \cos \alpha = k_0 \tan \alpha$$

(4.2)

$$k_0 = l_s \sin \alpha - \frac{\beta \cos^2 \alpha}{2 \omega} - \left[ \left( \frac{\beta \cos^2 \alpha}{2 \omega} - l_s \sin \alpha \right)^2 - \left( l_s^2 + \frac{\beta^2 y^2}{c^2 \cos^2 \alpha} \right) \right]^{1/2}$$

(4.3)

The group velocity direction is towards the center of the wavenumber circle. Thus, for an east-tilted boundary, the rays are initially directed slightly southwards.

Figure 6 shows graphically that the normal to the coast will not intersect the wavenumber circle at all if the orientation of the coastline is nearly east--west. From (4.2) we infer that free Rossby waves will only leave the boundary if

$$(2\omega y_0 / c \cos \alpha)^2 < 1$$

Thus, at any latitude north of the equator, the energy will be trapped next to the coast unless

$$|\alpha| < \cos^{-1} (2\omega y_0 / c)$$

(4.4)

Similarly, for latitudes with

$$|y_0| > (1/2) c/\omega$$

(4.5)

no free Rossby waves can exist for any slope of coasts. For the annual period this corresponds to about 70° from the equator, while for the semi-annual period the Rossby waves can extend only about 35° away from the equator.

In the Pacific the slope of the coast of Central America is about 40°; but in the Atlantic, along the Guinea Gulf, it is almost due east--west. For this latter coast, which is at 5°N, the condition (3.5) for the propagation of annual period waves is $|\alpha| < 84°$. Very little of the Guinea coast lies at an angle greater than 6° to the zonal direction. Thus, no free Rossby waves are possible from this coast.

No structure along the coast. If we consider initial conditions without structure along the coast, $l_s = 0$, and we have

$$k_0 = -\frac{\beta \cos^2 \alpha}{2 \omega} \left[ 1 - \left( 1 - \frac{4 \omega^2 y_0^2}{c^2 \cos^2 \alpha} \right)^{1/2} \right]$$

(4.6)

$$l_0 = -k_0 \tan \alpha$$

(4.7)

For a sloping coastline, (3.1--3.6) still apply, with the conditions (4.5), (4.6)
and
\[ x_0 = y_0 \tan \alpha \]  

(4.8)

The phase at a fixed time is given by the somewhat involved formula
\[
\phi = \left[ -\frac{k_0 c}{2\omega} \left( 1 - \frac{4\omega^2y_T^2}{c^2} \right) + \frac{\beta y_T^2}{c} \right] \Theta - \frac{\beta}{c^2} y_T^2 [\cos(\Theta + 2\Theta_0) \sin \Theta] \]  

(4.9)

Figure 7 shows a plot of the rays for a coastline inclined at \( \pi/4 \), demonstrating the tendency for a caustic to develop near the equator. For slopes of opposite sign the ray diagrams are simply the reflection about the equator of Fig. 7. The focus remains almost exactly at the same place as it did for an ocean with a meridional boundary. After a straightforward but lengthy calculation it can be shown that the leading order terms in a Taylor series expansion for the focus position, in powers of the angle \( \alpha \), are
\[
x = -\frac{\pi c}{4\omega} - \frac{c}{2\omega \pi} \sin^2 \alpha \]  

(4.10)

and
\[
y = -\frac{c}{\pi^2 \omega} \sin^3 \alpha \]  

(4.11)

Thus, there is a small displacement towards the west, and a smaller displacement to the north or south according to whether the ocean boundary tilts to the east or west.

Figure 8 shows where the energy is after a certain length of time. Near the eastern boundary there is a hatched region, in which at a given point there is "information" from several different times. Moreover, information after six years is to be found nearer the coast than after four years, say. Of course, this

![Fig. 7. Ray paths for a coastline sloped northeastward.](image-url)
diagram does not tell us where the energy comes from. This can be obtained from the ray diagrams (Figs. 3 and 7). The explanation of the above paradox is then obvious: the older information has come at a relatively slow group velocity from remote latitudes.

5. COMPARISON WITH MODAL THEORY

When the boundary conditions to (2.1) are separable, the theory of equatorial modes can be applied to the problem, and should provide consistent results. The case of a straight north—south coast furnishes such an example, and we will compare the two approaches, especially in regard to the focus and shadow-zone features of the ray solutions.

The separation of variables is made in the form

\[ u(x, y, t) = \sum_m a_m \psi_m(y') \exp[i(k_m x' - \omega' t')] \]  
(5.1)

where \( x', y' \) and \( t' \) are non-dimensionalized variables as used in the current literature (e.g. Moore and Philander, 1977):

\[ (x', y') = (x, y)(\beta/c)^{1/2}, \quad t' = t(\beta c)^{1/2} \]  
(5.2)

and \( \psi_m \) are the eigensolutions appropriate for the unbound equatorial \( \beta \)-plane:

\[ \psi_m(y) = \exp\left(-y^2/2\right) H_m(y)/(2^m m! \pi^{1/2})^{1/2} \]  
(5.3)
Fig. 8B. Time contours for annual period waves generated at a sloping coastline.
where \( H_m \) is the \( m \)th-order Hermite polynomial.

Using (5.1), equation 2.1 becomes a dispersion relation between \( \omega, k_m \) and \( m \):

\[
k_m = \{-1 \pm [1 - 4\omega'2(2m + 1)]^{1/2}\}/2\omega' \tag{5.4}
\]

The solutions are therefore specified completely by (5.1), (5.3) and (5.4), once the amplitude factors \( a_m \) are known.

To compare the results of Section 3 with the modal theory, we consider the westward propagation of energy from a limited portion of the eastern boundary:

\[
v(x = 0, y, t) = e^{-i\omega t}[\tanh(y' + 12) - \tanh(y' - 12)]/2 \tag{5.5}
\]

This forcing is then approximated by a finite eigenmode expansion, with a truncation limit of \( m = 100 \). Let

\[
v(x, y, t) = e^{-i\omega t}V(x', y') \tag{5.6}
\]

where \( V \) is a complex amplitude. Then

\[
V(x, y) = \sum_{m=1}^{100} a_m e^{ik_m x} \psi_m(y) \tag{5.7}
\]

Here, the \( m = 0 \) Yanai wave is dropped from the solution because it has only eastward energy propagation. The amplitude envelope of the waves will be given by \((VV*)^{1/2}\) and the real part of \( V \) will show the appearance of the

![Fig. 9. Solutions to (2.1) calculated using equatorial modes. The value at \( x = 0 \) was used to determine the spectrum of the eigenfunction expansion up to a truncation limit of \( m = 100 \). The real part shows the solution at \( t = 2n\pi \) years. The real part is contoured at \(-0.1 \pm n, n = 1, 2, 3\ldots\) avoiding the very small-amplitude, short-wave pattern in the shadow zone.](image-url)
field at \( t = 2k\pi, k = 0, 1, 2 \). Figures 9 and 10 show the two fields.

The wave energy in the shadow zone is less than 1% of that along \( x = 0 \), and the energy near the focus is 40 times greater. The waves which do enter the shadow zone have very short zonal wavelengths, and are associated with the low-amplitude, short-wave ripples involved in the approximation to the boundary condition. Ray theory would allow these waves to reach the shadow zone after a very long time because of the high value of \( \ell_0 \). The effect of damping is eliminated from the mode calculation.

The caustic can be seen in evidence as the narrow strip on the edge of the shadow zone where the amplitude has increased twofold over its initial value. In the real part of \( V \), a short-wave pattern is seen to lie along this strip. The solution at the equator is also disturbed near the focus, such that the long-wave nature of the equatorial response seen in the eastern portion of the basin is changed to short waves at about 100°W. This indicates a substantial modification of the equatorial response due to extra-equatorial wave energy.

**Focusing.** If the forcing causes a response with \( \nu(y) = 1 \) for \(-\infty < y < \infty\), then the modal amplitudes \( a_m \) are analytically known:

\[
a_m = \int_{-\infty}^{\infty} \psi_m(y) \, dy
\]

or

\[
a_{\pm n} = (2n!\pi^{1/2})^{1/2} / 2^{n-1/2}n!
\]

\[
a_{\pm n+1} = 0
\]  

(5.8)  

(5.9)

From Section 3, it is seen that the focus lies at \( x' = \pi/4\omega' \). At this particular
longitude the solutions are given by

$$V(x_f, y) = \sum_{n=0}^{\infty} a_{2n} \psi_{2n}(y) \exp[i(k_{2n}\pi/4\omega')]$$  \hspace{1cm} (5.10)$$

Now, if \( \omega'2m \ll 1 \), the dispersion relation may be approximated by

$$k_m = \omega(2m + 1)$$  \hspace{1cm} (5.11)$$

To the approximation that the forcing extends to infinity, and that (5.11) holds for the wavenumber \( k_m \), the solution at the focal longitude is

$$V(x_f, y) = \sum_{n=0}^{\infty} a_{2n} \psi_{2n}(y) \exp[i(n\pi + \pi/4)]$$

$$= e^{i\pi/4} \sum_{n=0}^{\infty} a_{2n} (-1)^n \psi_{2n}(y)$$  \hspace{1cm} (5.12)$$

From the properties of the Hermite functions (Abramowitz and Stegun, 1964) it is found that

$$(-1)^n a_{2n} = \pi^{1/4}(2n!)^{1/2}(-1)^n/2^{n-1/2}n! = (2\pi)^{1/2}\psi_{2n}(0)$$  \hspace{1cm} (5.13)$$

That is,

$$V(x_f, y) = e^{i\pi/4}(2\pi)^{1/2} \sum_{n=0}^{\infty} \psi_{2n}(0) \psi_{2n}(y)$$  \hspace{1cm} (5.14)$$

This exactly matches a delta function at \( y = 0 \) of amplitude \( e^{i\pi/4}(2\pi)^{1/2} \).

6. RAYS FOR THE WORLD OCEANS

When the coast is non-meridional and the wind has structure, the ray paths become more difficult to develop analytically. However, if the coastline shape is treated as slowly varying, we may follow the rays by using a series of locally defined \( \beta \)-planes to integrate the ray paths analytically. We use

$$\omega = -\beta(\theta) k/(k^2 + l^2 + f^2(\theta)/c^2)$$  \hspace{1cm} (6.1)$$

$$f = 2\Omega \sin \theta, \quad \beta = (\partial f/\partial \theta)/a$$  \hspace{1cm} (6.2)$$

$$d\lambda/dt = (a \cos \theta)^{-1} \partial \omega/\partial k$$  \hspace{1cm} (6.3)$$

$$d\theta/dt = a^{-1} \partial \omega/\partial l$$  \hspace{1cm} (6.4)$$

Figure 11 shows the rays of annual period for the world oceans. The initial meridional structure of the waves is that \( l_0 = \pm 1.4 \times 10^{-6} \text{ m}^{-1} \). The shape of the coast at the start of a ray is approximated by a straight line at an angle from the meridional. While the Atlantic is considerably narrower than the
Fig. 11. Ray paths for annual period waves in the world oceans. Equations 5.2–5.7 were used to trace the rays $c = 3.13 \text{ m s}^{-1}$. The boundaries were approximated as straight-line segments inclined from due north.

Pacific, substantial turning still takes place. Also, the focus of rays from the South Pacific is particularly sharp.

The rays traced on the sphere are seen to have the same character as those developed on the equatorial $\beta$-plane. This indicates that the important turning effect is not the sphericity of the Earth per se, but rather its influence on the internal radius of deformation. This feature of the present analysis distinguishes these rays from those investigated for the atmosphere by Hoskins (1978) and Karoly (1978).

7. EFFECTS OF MEAN CURRENTS

7.1. Introduction

The previous sections dealt with Rossby waves in an ocean with no mean currents; here, the influence of such currents on the previous theory is examined. Intuitively, one might think that if the phase speed of the Rossby waves is larger than the mean currents (roughly a few cm s$^{-1}$), the waves will not be significantly affected. This state of affairs would apply in low latitudes where phase speeds are about 30 cm s$^{-1}$. At high latitudes where phase speeds are nearer 2–3 cm s$^{-1}$ (comparable with mean currents) effects might be more drastic. In general, mean currents are a function of depth as well as latitude and longitude. Although analysis to cope with such a complexity is beyond the scope of this paper, analysis of some simpler problems can provide some insight.

7.2. Barotropic zonal mean current

To quantify the matter of phase speeds matching current speeds we consider a model with a barotropic mean current $U$, independent of zonal position $x$ but varying with latitude. The dispersion relation is the same as (2.2)
except that the impressed frequency $\omega$ is now replaced by the Doppler-shifted frequency $\omega_D = \omega - Uk$, i.e. (2.2) is modified to

$$\omega - Uk = -\beta k/\left(k^2 + l^2 + f^2/c^2\right) \quad (7.1)$$

As before, the medium is time-independent, and since $U$ is only a function of $y$, it also is zonally uniform. This means that the zonal wavenumber and frequency are again conserved along a ray.

$$\frac{dx}{dt} = \frac{\partial \omega}{\partial k} = \frac{\omega_D}{k} + 2\omega_D^2/\beta + U = \omega/k + 2(\omega - Uk)^2/\beta \quad (7.2)$$

$$\frac{dy}{dt} = \frac{2\omega_D^2}{\beta} \left(\frac{\beta k}{\omega_D} - k^2 - \frac{\beta^2 y^2}{c^2}\right)^{1/2} \quad (7.3)$$

These equations suggest that a ray path will be strongly affected if $dy/dt \to 0$. This can occur when $\omega_D = 0$, i.e. $\omega - Uk = 0$ (a critical layer) or when

$$-\beta k/(\omega - Uk) - k^2 - \beta^2 y^2/c^2 = 0 \quad (7.4)$$

This latter condition (for reflection) is usually less severe than that for the critical layer condition.

A critical layer can occur for a westward current when $\omega/k = U$. For a current of strength $U$, absorption will occur for rays starting at the latitude $y_0$ such that

$$k = -\frac{\beta}{2\omega_D(y)} \left[1 - \left(1 - \frac{4y_0^2\omega_D^2(y)}{c^2}\right)^{1/2}\right] \quad (7.5)$$

For a current of 5 cm s$^{-1}$, this is typically 27° N. For rays starting at a higher latitude the radical in (7.5) is negative and so, for these rays also, absorption will take place. This simple analysis suggests that the main effect of a westward barotropic mean current is to lead to absorption of rays coming from higher than some critical latitude. For a mean flow of 5 cm s$^{-1}$ this can be a moderately low latitude as the above example shows (about 30°), but for lower current speeds (2 cm s$^{-1}$) is a rather higher latitude (about 45°).

The second condition whereby a zero in $dy/dt$ can be obtained is when (7.4) is satisfied. This occurs when

$$U = \omega/k + \beta/(k^2 + \beta^2 y^2/c^2) \quad (7.6)$$

For a zonal current at the equator to reflect a ray from 10° N needs an eastward current of approximately 100 cm s$^{-1}$. However, a mean current of 30 cm s$^{-1}$ located at 10° N can reflect energy from anywhere poleward of 20° N. These values are ridiculously high for a barotropic current though surface currents of this amplitude are possible (the counter current for example) and these might play some role. Before considering this aspect, however, we will consider more realistic barotropic currents.
7.3. Barotropic Sverdrup mean currents

In this section we will use a more realistic, self-consistent mean current distribution given by the Sverdrup balance. Here, meridional \( (V) \) and zonal \( (U) \) components of the mean flow exist. For a wind stress \( (\tau^x, \tau^y) \) say, mean currents satisfying
\[
-fV = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + X \tag{7.7}
\]
\[
fU = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + Y \tag{7.8}
\]
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{7.9}
\]
result, where \( X = \tau^x / \rho_0 H \) and \( H \) is the ocean depth, taken to be uniform.

The inclusion of mean currents leads to a Doppler shifting of the frequency \( \omega_D = \omega - kU - lV \) and so the dispersion relationship (2.2) becomes
\[
\omega_D = \omega - Uk - Vl = -\beta k / (k^2 + l^2 + f^2/c^2) \tag{7.10}
\]
where \( U, V \) are given by (7.7–7.9).

Because \( U \) is a function of zonal coordinate \( x \), the wavenumber is no longer conserved along a ray, so we need an equation governing the evolution of \( k \) (Whitham, 1960):
\[
\frac{dk}{dt} = -\frac{\partial \omega}{\partial x} = -k \frac{\partial U}{\partial x} \tag{7.11}
\]
The ray equations are
\[
\frac{dx}{dt} = \frac{\partial \omega}{\partial k} = U + \omega_D / k + 2\omega_B^2 / \beta \tag{7.12}
\]
\[
\frac{dy}{dt} = \frac{\partial \omega}{\partial l} = V + 2l\omega_B^2 / \beta k \tag{7.13}
\]
In principle, with the dispersion relation (7.10) these form a closed set of four equations for the four unknowns along the ray, \( k, l, x, y \). However, since the equations have to be integrated numerically we replace (7.10) by
\[
\frac{dl}{dt} = -\frac{\partial \omega}{\partial y} = -\left( k \frac{\partial U}{\partial y} + l \frac{\partial V}{\partial y} + \frac{2\beta^3 ky/c^2}{(k^2 + l^2 + \beta^2 y^2/c^2)^2} \right) \tag{7.14}
\]
A new feature of these equations is the more complicated expression for the meridional position of the ray. Since the ray velocities in this direction can be quite small, even a modest meridional velocity can be significant.

Solutions to (7.11–7.13) were obtained with \( X = X_0 \cos(my + \theta) \) for the case \( X_0 = \tau / \rho_0 H = 0.25 \times 10^{-5} \) cm s\(^{-2} \) which corresponds to a wind stress of 1 dyne cm\(^{-2} \) blowing on an ocean 4000 m deep; \( m = 10^{-8} \) cm\(^{-1} \) for \( \theta = 0 \) and \( \theta = \pi/2 \). The ray paths are plotted in Figs. 12a, b. Although the zonal mean velocities are much larger than the mean meridional velocities, the changes in the ray paths result from the meridional currents, also. The mean gyres which
Fig. 12. Above: ray paths in the presence of mean barotropic Sverdrup currents. The corresponding mean flow is shown in Fig. 13a. Below: ray paths in the presence of Sverdrup currents. The corresponding mean flow is shown in Fig. 13b.

the wind stress induces are drawn schematically in Fig. 13. Comparing Fig. 12 with Fig. 3 for no mean flow we see that for rays originating equatorward of $15^\circ$ N, there is little change in the ray paths, particularly for $\theta = 0$ where the low-latitude currents are primarily zonal and much less than the Rossby wave speed.

Further north where the Rossby wave speeds are lower, the rays are more strongly influenced by the mean currents. To assess the relative importance of the mean $U$ and $V$ terms, runs were made in which the $V$ terms in (7.13) and (7.14) were suppressed. The rays were modified then only by the $U$ fields. By comparing this pattern with the $U = V = 0$ case and the correct version of (7.23–7.26) it was clear that both $U$ and $V$ were important in the physically
expected sense. $V$ is in fact very small for the Sverdrup flow; for example, a wind stress of 1 dyn acting on an ocean of depth 4000 m will induce a maximum meridional flow of only a few mm s$^{-1}$. However, because the wave velocity along a ray in this direction is also small, this can have a strong effect it is not sufficient to consider only the larger zonal mean currents. Of course, as Fig. 12 shows, rays which start sufficiently far north will in due course
move through areas where the mean flow is predominantly zonal and also through areas where the mean flow is predominantly meridional. In such cases the rays will be influenced by both types of current. For example, within the 10-year period of Fig. 12, the ray from 30° N is subject to both types of displacement. Low-latitude rays tend to be influenced more by only one type of displacement. For higher-latitude rays also, within the span of 10 years, this is true, though if the calculation were continued longer the latter rays would experience more variety.

A consequence of the inclusion of a mean current is that the focus is less sharp than it was in its absence. However, rays emanating equatorward of 15° N do still focus, particularly for θ = 0 when the focus is on the equator. There is less evidence for a caustic, however, in this case.

The overall effect of mean currents is not catastrophic. Figure 12 bears a reasonable resemblance to Fig. 3. The main difference is quantitative. A reasonably realistic mean wind stress was used to generate the mean currents. This gives rise to quite weak barotropic currents. If the currents are stronger, the effects will be substantially greater.

7.4. Baroclinic mean currents

In the ocean mean currents are not barotropic, the surface flow usually being much stronger than the deep flow. The interaction between baroclinic waves and baroclinic mean currents cannot in general be easily determined.

A simple model which can illustrate some potentially interesting features of this process is one in which the ocean is considered to consist of two layers. The stronger surface currents are confined to the upper layer. If the model consists of a uniform upper layer of depth h and density ρ₁ overlying a deep inert layer of density ρ₂ the model equations are those of a reduced gravity model:

\[
\begin{align*}
\partial u/\partial t + u \partial u/\partial x + v \partial u/\partial y - fu &= -g' \partial h/\partial x + X/H \quad (7.15) \\
\partial v/\partial t + u \partial v/\partial x + v \partial v/\partial y + fu &= -g' \partial h/\partial y \\
\partial h/\partial t + \partial (hu)/\partial x + \partial (hv)/\partial y &= 0 \quad (7.17)
\end{align*}
\]

where \( g' \) is reduced gravity: \( g' = g \Delta \rho/\rho \). If we consider perturbations on a mean flow \( U, V \) the equations governing the perturbations are

\[
\begin{align*}
\partial u'/\partial t + U \partial u'/\partial x + V \partial u'/\partial y - fu' &= -g' \partial h'/\partial x \quad (7.18) \\
\partial v'/\partial t + U \partial v'/\partial x + V \partial v'/\partial y + fu' &= -g' \partial h'/\partial y \\
\partial h'/\partial t + \partial (Uh')/\partial x + \partial (Hv')/\partial x + \partial (Hu')/\partial y &= 0 \quad (7.20)
\end{align*}
\]

In (7.18–7.20) variations of the zonal flow in the meridional direction have been retained. This is in contrast to Section 7.2 where such variations were disregarded. The rationale is as follows. Within our current understanding of wind-driven circulation, mean flows are on a scale comparable with that of
the wind. This is large, so the approximation that the mean currents vary on a scale large compared with the wave scale is generally good. However, surface currents can vary on a shorter horizontal scale. For example, the counter current located at about 5° N, has a scale of only 1 or 2° of latitude, is strong, (about 30 cm s⁻¹) and surface-intensified. The variation in the thermocline depth across the counter current trough is quite large. For example, the depth of the 14°C isotherm changes from 100 m at the trough to 200 m 3° to the north or south. This suggests that, within the framework of the model, depth fluctuations are important. Anderson and Killworth (1979) showed that when this variation in the mean depth is comparable to the mean depth the main effect on the Rossby wave propagation speed resulted not from effects of mean currents but from variations in the layer depth.

It is convenient to work in terms of $Hu$ and $Hv$; from (7.18) and (7.19) we obtain, by differentiating with respect to time and substituting from (7.19) and (7.18) respectively,

$$
\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 H_u + f^2 H_u = -fg'H \frac{\partial h}{\partial y} - g'H \frac{\partial^2 h}{\partial x \partial t} - Ug'H \frac{\partial^2 h}{\partial x^2} \tag{7.21}
$$

$$
\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 H_v + f^2 H_v = +fg'H \frac{\partial h}{\partial x} - g'H \frac{\partial^2 h}{\partial y \partial t} - Ug'H \frac{\partial^2 h}{\partial x \partial y} \tag{7.22}
$$

The first terms in the above equations can be dropped since we are interested in low-frequency motion (i.e. motion on a time long compared with $f$). Substitution for $Hu$, $Hv$ into (7.20) then yields

$$
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (Uh) - \frac{g'H}{\beta^2 y^2} \frac{\partial h}{\partial y} + \left(\frac{g'H}{\beta^2 y^2}\right) \frac{\partial h}{\partial x} - f^2 \left(\frac{\partial^3}{\partial x^3} - \frac{\partial}{\partial y} \left(f^2 \frac{\partial^2 h}{\partial x \partial y}\right)\right)
$$

$$
\frac{\partial}{\partial y} \left(f^2 \frac{\partial^2 h}{\partial x \partial y}\right) = 0 \tag{7.23}
$$

Scale analysis suggests that the important terms in (7.23) are the first four. This shows that variations in the ray paths result from variations in the thermocline depth in a fairly innocuous way, i.e. the waves will move faster where the thermocline is deep and more slowly where it is shallow. In this model there is no "effective $\beta$" term resulting from variations in $H$ (dynamic topography), as there is in the topographic Rossby wave problem. This is because this term is exactly cancelled by the $Uh_x$ term in the continuity equation if $U$ is assumed to be geostrophically balanced by variations in $H$.

8. SUMMARY

This analysis has shown that one must be careful in applying the non-dispersive approximation to low-frequency Rossby waves. For annual period waves, the turning of the fastest internal modes will be significant and will seriously affect the energy propagation, even within basins of the size of the
Atlantic. This turning is largely due to variations of the wave speed with latitude caused by changes in the internal radius of deformation, whereby high meridional wavenumbers are generated, causing meridional components of the group velocity.

It has been seen that waves excited from an eastern boundary with no meridional structure will have an almost exact focus at a distance $\pi c/4\omega$ from the coast. While the $l_0 = 0$ component may be only a small part of the overall response to typical winds, the strong forcing may locally exaggerate the response near the focus out of proportion to its excitation. The westward-propagating equatorial modes are seen to match that structure obtained using this $\beta$-dispersion on the equatorial $\beta$-plane, a result consistent with the view that the modes must arise due to the development of a standing wave pattern through reflections. Since the wave ray paths can be calculated for any slowly varying coastal shape, the restriction of separable boundary conditions imposed on the equatorial modes analysis can be relaxed. This allows an improved understanding of the response of particular ocean basins to annual wind variations.

The influence of mean currents on the energy propagation occurs through several mechanisms. The barotropic zonal flow can induce critical layers or enhance the turning. While abnormally high barotropic zonal velocities are needed to keep the wave energy from crossing the equator, an eastward current will move the turning latitude equatorward, a westward current poleward.

Barotropic meridional currents, although they may be quite small, can have noticeable effects on the ray paths, especially near the turning latitude, where the meridional component of group velocity is very small.

For baroclinic mean currents, the effect on the ray paths occurs by speeding up the waves where the thermocline is deep and slowing them down where shallow, without appreciable turning.

This analysis has attempted to point out the directional behavior of the low-frequency Rossby wave energy, and to explain some features of numerically simulated circulation patterns. The consideration of energy modulation and the details of the wave excitation are problems which must be treated before this technique may be applied to prognostic calculations of the response in complex basins to winds with complex structure.

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